ON CAYLEY'S THEORY OF THE ABSOLUTE.

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In the following pages I attempt to show, as a matter of purely pedagogic interest, how simply and naturally Cayley's theory of the Absolute follows from a small number of very elementary geometrical conceptions, without any appeal to analytical geometry. Where assumptions are made, the fact is frankly stated; the few points where more advanced mathematical reasoning is needed for the actual proof are clearly indicated; my contention is not that every step in the rigorous proof can be presented under the guise of elementary mathematics, but that it is quite possible to develop the theory so as to be intelligible and interesting to average students at a much earlier stage than is customary.

Let any simple diagram be drawn on a sheet of paper, *e. g.*, a circle with a straight line cutting it in two points. Let this sheet of paper be held at some distance away, in such a position as to be slightly oblique to the line of sight. A difference will now present itself, of such a nature as to suggest that the properties of the figure are of two distinct kinds. It will be as evident as before that there is a straight line, and a curve cut by the line in two points; but it will not be perfectly evident that the curve is a circle, it will appear as an oval curve. Similarly, if we have two straight lines intersecting at right angles, the fact that there are two intersecting straight lines will be evident under whatever aspect the figure may be viewed, but the angle between them will not appear to be a right angle. The same effect will be observed if, keeping the diagram fixed, the position of the eye be changed. Thus we see that the properties of a plane figure are of two distinct kinds; some are purely relative, dependent on the point of view; others are more intimately connected with the figure itself, they have no relation to the point of view. The effect of changing the point of view is considered in the mathematical theory of projection, which must now be briefly explained.

Given any figure in a plane (1), and a point V not in this plane, let V, the centre of projection, be joined to all the points of the given figure, A, B, C, etc., and let the points in which these joining lines cut a second plane (2) be denoted by A', B', C', etc. To an eye at V, with no

power of determining the actual position of a distant plane, the two figures would be indistinguishable; figure (2), in its proper place, might be substituted for figure (1) without the change being discovered. And yet in many ways the two figures if drawn side by side on one sheet of paper would appear very different; one might contain a circle, a pair of straight lines at right angles, a number of parallel lines, while in the other figure no one of these things could be found. It will be seen that the properties in which any difference is to be observed in the two figures all depend on measurement of lines or angles; these properties are called metric; the properties that are unaffected by the change from one figure to the other, unaffected, that is, by projection, are called projective.

Starting with any figure, (1), we may project from any selected centre, thus obtaining a projection, (2); taking any new centre of projection, and from this projecting (2) on to any plane we obtain a figure (3). This process can be repeated as often as we please, and after any number of projections we arrive at a figure which, while not strictly a projection of the first, in the sense just given to the word, is still in a projective relation to it; the projective properties, being unaffected by every projection separately, are unaffected by the series of projections, and thus the initial and final figures agree as regards all projective properties.

By the definition of projection a point becomes a point. Also a straight line becomes a straight line; for if points A, B, C, etc., lie on a straight line, a simple application of Solid Geometry shows that their projections lie on a straight line. Moreover, straight lines that meet in a point project into straight lines that meet in a point. These laws may be formulated as follows:—points and straight lines become points and straight lines, and properties of collinearity and concurrence are unaltered by projection. These are the general laws; to these we admit no exceptions whatever.

Certain apparent exceptions must be investigated. Let the plane through the centre V parallel to the plane (2) cut the plane (1) in the line YZ. The line joining V to any point X on YZ is parallel to the plane (2), and accordingly, in the Euclidean phraseology, never meets it; that is, the point X has no projection. This however being at variance with our general laws, is not an admissible form of speech; we have to find some other mode of expressing the idea. Take in the plane (1) any line through X; let a point A move along this line, approaching X; the projection of A recedes more and more, and when A moves up to X the

point A' is certainly not at any finite distance; we express this by saying that the projection of the point A is at infinity. This we may regard as a conventional mode of speech expressing the same idea as the Euclidean "never meet"; geometrically we have postulated the existence of a special set of points, the points at infinity, for the sake of making our general laws hold without exception. As regards these points we have at once the theorem, all points at infinity lie on a straight line, the line at infinity; for the points at infinity are by definition simply and solely the projections of the points on the line YZ, and by our general laws the projection of this straight line, like that of any other, is a straight line.

Now consider any two or more lines, AX, BX, that intersect at X, on the line YZ. The point X projects to infinity, and thus the lines in the projection have their common point at infinity; now the planes VAX, VBX, etc., all contain the line VX, parallel to the plane (2), and consequently the lines $A'X'$, $B'X'$, etc., in which they are cut by this plane (2) are parallel. We are thus led to the conclusion that parallel lines meet at infinity, and this again is merely another way of stating the idea involved in the Euclidean phrase "never meet". Thus we see that while the law, concurrent lines project into concurrent lines, is not interfered with, the concurrent lines may happen to be parallel. The property of concurrence is projective; the property of parallelism is metric. The first is unalterable by projection; the second may be destroyed.

We now consider the effect of projection on the separate points that lie on any one line. These being A, B, C, etc., we require only the lines VA, VB, etc., and the line in which the plane of these is cut by the plane (2); hence the whole diagram now lies in the plane $VABC$; we may even speak of projecting from the one line on to the other. It is at once apparent that by suitably chosing the position of V, the distance between two points can be altered to any extent; and further, taking three points A, B, C, we can project so that the distances apart shall become anything we please; for, taking any line through A, and measuring on that the distances Ab, bc, that we wish to be assumed by AB, BC, all that is necessary is to use the intersection of Bb, Cc, as the centre of projection. This is usually stated in the form:—any three collinear points can be projected into any three collinear points. To project A, B, C into A', B', C', join AC', use any point on the line CC' as centre of projection, so obtaining on the auxiliary line the points $Ab'C'$; now

use the intersection of bB', AA' as a new centre of projection, and project AbC' on to the second line. Thus the three points A, B, C are projectively related to the three A', B', C'; in the more extended sense of the term, the one figure is a projection of the other. Our conclusion is therefore that three points on a line have no projective relation other than that of collinearity.

Passing on to consider four points on a line, we are led to the important result that four points have a projective relation. If we take four points A, B, C, D on a line, and four points A', B', C', D' on another line, and attempt to connect them projectively, we proceed as before for the three points A, B, C, but then the point D on the first line gives us a definite point on the auxiliary line, and this gives a point D'' on the second line; it can be shown that D'' will be the same however the arbitrary elements in the construction may be varied; we have then no control over the position of this point D'', we have no way of making it come at D'. The truth is more obvious if the two triads A, B, C and A', B', C' are in projective position; every point D on the first line gives as its correspondent on the second line a perfectly determinate point D'. Hence four points on a line have some relation that is unalterable by projection. For our present purpose it is not essential that we should know the precise nature of this relation; it is enough to assure ourselves that it must be expressible by some number. It cannot be any length, nor any product of lengths, for these would certainly be altered by projection from a line on to any parallel line; it is consequently some number that can be obtained without measurement, and this number is not even a simple ratio of lengths, for we have seen that two lengths can be made to assume any desired magnitude.

There is one particular arrangement of the four points that is of special importance. Let four lines be taken, then every one intersects every other, thus giving six points of intersection, these falling into three pairs that are not joined; draw the three joining lines. The figure thus constructed is a complete quadrilateral; the four lines by which it is determined are the sides, the six intersections of these lines are the vertices, and the three lines drawn to complete the joining of the vertices are the diagonals. On any diagonal we have two pairs of points, viz., two vertices and two intersections with the remaining diagonals; these two pairs are said to be harmonic. Thus two pairs of points on a line are harmonic when a quadrilateral can be described with

vertices at one pair, and diagonals through the other pair. The whole diagram, depending simply on points and straight lines, with properties of collinearity and concurrence, is plainly projective; hence, if one of the two pairs be given, and one point of the other pair, the fourth point, the harmonic conjugate to P with respect to A, B is determined. Moreover, if A, B are the pair at which the vertices lie, and P, Q the pair through which the diagonals pass, it is easy to construct a quadrilateral having vertices at P, Q and diagonals through A, B; this proves that the harmonic relation involves the two pairs symmetrically, whatever relation the pair A, B has to the pair P, Q, that same relation has the pair P, Q to the pair A, B. Not only this, but also the two points of either pair are involved symmetrically; this can be shown in the usual manner, by showing that $ABPQ$ can be projected into $BAPQ$. Hence when two pairs of points are known to be harmonic, not only the order of the pairs, but also the order of the points in a pair, is a matter of absolute indifference. Similarly, two pairs of concurrent lines may be harmonic, this being the case if the pairs of points in which they are cut by any transversal are harmonic.

With the help of the idea of parallelism, the harmonic relation leads at once to the idea of the bisection of a line. For let the quadrilateral be a parallelogram, then seeing that one of the three diagonals is entirely at infinity, the point Q on one of the remaining diagonals is at infinity. But in ordinary geometry the two diagonals of a parallelogram are said to bisect one another, that is, AB is bisected at P; hence in the modified phraseology here adopted, instead of saying that AB is bisected at P, we say that AB is divided harmonically at P and infinity.

This conception enables us to mark a scale of measurement on any line. Let two points O, Z, be arbitrarily assumed, and assign to these any arbitrarily selected numbers, *e. g.*, 0 and ∞; take any point A, and assign to this the number 1; take the harmonic conjugate to O with respect to AZ, call it B, and assign to it the number 2; the harmonic conjugate to A with respect to BZ is to be called C, and marked with the number 3, and so on. The harmonic conjugate to Z with respect to OA is the point $\frac{1}{2}$, the harmonic conjugate to Z with respect to this and O is $\frac{1}{3}$, and so on. Thus every point has a numerical magnitude associated with it; when the point Z is actually the point at infinity on the line, the points 0, 1, 2, 3, etc., are all at equal distances apart; the number associated with a point measures its distance from O, the length OA being the unit of measurement.

But though this gives a scale on the line, that is, a means of measuring on the one line, it gives no way of comparing measures on different lines; for even if we take the intersection of the two as the point O, and suppose the line at infinity, and consequently the points at infinity on the two lines, to be given, we have still no way of comparing the selected units, OA and OA', on the two lines.

Similarly with the help of the idea of perpendicularity, the harmonic relation gives us the idea of equality of angles about a point, and if we can draw at a point a line perpendicular to any line through the point, then we can mark off a scale of angular measurement about that point. For let there be given two lines o and a; through their intersection draw a line a' perpendicular to a, and let b be the harmonic conjugate to o with respect to the pair a, a'; then the angle oa is equal to the angle ab, as can be seen at once by drawing any transversal perpendicular to a. We can then draw through the point a line that shall make the same angle with b, and so on, and thus construct a scale of angular measurement about the point.

But just as before, this gives no way of comparing angles about two different points.*

We cannot get any further in measurement unless we take some postulate, *e. g.*, that of the existence and possible construction of a circle. For we cannot obtain this as the locus of points at equal distances from a fixed point unless we define equal distances; nor as the locus of the vertex of a right angle whose sides pass through fixed points unless we assume that angles between parallel lines are equal, and so assume that we can determine the equality of angles about different points. If however the existence and construction of a circle with any centre and any radius be postulated, we can use this as a means of defining equality of lines in different directions, and, if we choose, of defining right angles. We can then deduce all properties and constructions for perpendicular and parallel lines—all the constructions of metric geometry, in fact—with very little alteration of the arrangement of the propositions in Euclid's elements, which is most literally the geometry of the straight line and circle.

Now the circle projects into a curve that is not a circle,

* One caution should be given here. It might be imagined that we could compare angles about different points by means of parallel lines, and lines in different directions by means of diagonals of a rectangle; but it must be remembered that we have not yet got so much as a definition of equality for lengths on different lines or angles about different points.

though agreeing with it in the properties that a straight line can be drawn to meet it in two points but not in more, and that from a point it may be possible to draw two tangents, but not more; the special properties of a circle are all metric. Thus all metric properties and constructions depend on the circle, though some are more simply expressed with reference to the ideas of parallelism and perpendicularity.

We now pass on to see whether it is possible to construct any system of measurement that shall not have any dependence on these ideas, that is, any truly projective system. This we consider in the first place as relating only to points on a line.

We have seen that we cannot get any projective relation unless we use four points; in order to obtain any relation between O, P, we require other two points A, B, and then we have some number belonging to the set of four points (OP, AB). But here there is a certain lack of definiteness. If there be a number belonging to the four points, then any function of the number is itself a number equally belonging to the points, *e. g.*, the logarithm or the square of the first mentioned number; some one of this set of numbers must be selected. Also, the points A, B, C, D, are not projectively the same as B, A, C, D, hence the number must depend in some way on the order of the points.

Supposing for the moment that these two things, the form of the function and its dependence on the order of the points, have been decided upon, then if three points O, A, B, are arbitrarily chosen, the position of any point P on the line is absolutely determined by the value of this numerical mark. Bearing in mind that what we ordinarily do with points on a line is to measure their distances apart, position on the line being assigned by distance from any selected point, *e. g.*, the point O, we see that if we adopt the order O, P, A, B, in determining the value for the point P, we must adopt the order O, Q, A, B, in determining the value for the point Q. Moreover, if we are to obtain a result that can be interpreted as referring to the distance OP, even by a convention, we must regard O, P as points whose relation with respect to A, B has to be given. Hence we shall write the function as $f(OP, AB)$. Thus the mark which determines the position of the point P with regard to O, with respect to the points A, B, is $f(OP, AB)$; and similarly the mark for Q is $f(OQ, AB)$. By parity of reasoning the mark which determines the position of Q with regard to P is $f(PQ, AB)$.

The most important property of ordinary distances is that expressed by the relation $OQ = OP + PQ$; in order that

the marks now under discussion may be susceptible of simple interpretation in terms of distances, we must therefore have $f(OQ, AB) = f(OP, AB) + f(PQ, AB)$. This is found to be enough to determine, except as to a multiplier, which particular number out of all possible ones is to be selected; assuming this to be the case, we have arrived at the conclusion:—Instead of measuring the distance of a point P from a point O in the usual way, this giving a relation of the points that has no permanence in projection, we can assign a numerical mark, entirely unalterable by projection, expressing the relation of P to O with respect to an arbitrarily selected pair of points on the line. These two points being once selected, the mark belonging to any point P on the line with reference to O is called the generalized distance OP, and is denoted by the symbol $\overline{OP}$. Generalized distances on a line obey the same law as ordinary distances, viz, $\overline{PQ} + \overline{QR} = \overline{PR}$.

Similarly we can measure angles about a point if we have a pair of lines a, b through the point as a standard of reference. The generalized angular distance from any line p to any line q, that is, the generalized measure of the angle made by q with p, is $f(pq, ab)$, *i. e.*, $\overline{pq}$. And just as in the case of the linear measurements, these angles are subject to the law expressed by the equation $\overline{pq} + \overline{qr} = \overline{pr}$.

Thus we can construct a system of measurement on a line if we have on that line a certain absolute configuration, two fixed points; and a system of measurement about any point if we have an absolute configuration, two fixed lines through the point. In attempting to apply this to a plane, we require an absolute configuration that shall give us two points on every line, and two lines through every point. It is necessary that these be given by some configuration, for the number of lines in the plane being indefinitely great, it would not be possible to assign the two desired points separately for every line, and similarly as regards the points. Now the only configurations that can exist contain either a finite number of points and lines—which would not supply us with the necessary two points on every line, two lines through every point—or an indefinite number of points and lines. This last is therefore the one to be discussed. We confine ourselves for the moment to the points that have to be determined. We have nothing to do with random assemblages of points, for we require a configuration that can be *given*; hence the points must be given in one of two ways; as lying on some line, straight or curved, or

as lying in some area. This last is plainly inapplicable, for it would give an indefinite number of points on any line passing through the area; hence the desired configuration must be a line, straight or curved. The only essential property of this line is that it be met by a straight line in two points; this suggests the circle, or rather a projection of a circle. Now it can be proved that the only curve possessing this property of meeting a straight line in not more than two points is a projection of a circle; that is, a conic section, or a conic.

But it may be argued that this does not necessarily give two points on every line, for some lines do not meet the curve at all. But just as in algebra it is found convenient to say that a quadratic equation has always two roots, though these may be equal or imaginary, this convenience presenting itself in the general consistency thereby obtained in the results, so in geometry it is convenient and legitimate to say that a straight line always meets a circle in the same plane with it in two points, though these may be coincident (when the line is a tangent), or imaginary (when the line does not visibly intersect the circle). That is, just as in algebra we postulate the existence of imaginary quantities, so in geometry we postulate the existence of imaginary points.* Hence any projection of a circle does give us the necessary two points on every line to serve as a standard of reference for a system of measurement on that line.

Furthermore, from any point there can be drawn two tangents to a circle; even if the point be within the circle, so that no real tangents can be drawn, yet we say that there are two, the existence of the imaginary ones being postulated exactly as in the case of the points. The same thing holds therefore of any projection of a circle; and we are led to the conclusion that if any conic be given in a plane, the points and lines (tangents) provide a framework, a universal standard of reference, with respect to which metric relations can be projectively formulated. This conic is called the Absolute.

Since all metric properties are expressible as relations of the figure to the Absolute, the properties of parallelism and

* By the number of solutions of a geometrical problem is to be understood the greatest number that can be obtained when the data are arranged at pleasure in accordance with the given conditions; if a different arrangement of the data gives apparently a smaller number, the difference gives the number of imaginary solutions. Thus imaginary points and lines are postulated; they are the missing solutions of geometrical problems; the justification for introducing them is the generality and consistency thereby gained.

perpendicularity can be thus expressed. There are two particular relations of straight lines with respect to the Absolute that call for investigation, and it is found that one of these leads to the conception of parallelism, the other to that of perpendicularity. In the first place, two lines p, q, may meet on the Absolute. In this case, the tangents that can ordinarily be drawn from the point of intersection of the two lines coincide; and it is found that the particular function* adopted for the expression of the numerical mark has in this case the value zero. In metric geometry, when the angle between two lines is zero, the lines are said to be parallel; accordingly the system we are constructing will agree with ordinary geometry if we define parallel lines as lines that meet on the Absolute. The other relation of two straight lines is that expressed by the term conjugate; it is shown in elementary geometry that taking lines through a point P, cutting a circle in U, V, the locus of Q, the harmonic conjugate to P with respect to U, V, is a straight line, the polar of P; and that this is the chord of contact of tangents from P. Harmonic properties being projective, this holds for conics. The defining property of polars shows that if the polar of P pass through Q, then the polar of Q passes through P; the poles P, Q, as also their polars p, q, are said to be conjugate; and it is at once seen that conjugate lines are harmonic conjugates with respect to the two tangents a, b, that can be drawn from their intersection, and hence that (pq, ab) is projectively the same as (qp, ab). Applying this, we have $f(pq, ab) = f(qp, ab)$, *i. e.*, $\overline{pq} = \overline{qp}$, which states that the angle from q to p is equal to the angle from p to q. If we were speaking of angles in the ordinary sense, this would mean that the adjacent angles are equal, that is, the lines p and q would be said to be perpendicular; hence in the present system we define perpendicular lines as lines that are conjugate with respect to the Absolute. Hence one perpendicular to any given line can be drawn through any point T, by joining T to the pole of the line.

It has now been shown that with the help of the Absolute, we can construct a system of metric geometry; and the question arises, is there any way of choosing the conic, the Absolute, so that this system shall be the ordinary Euclidean geometry?

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* The function for (PQ, AB) is $k \log\left(\frac{AP}{PB} \Big/ \frac{AQ}{QB}\right)$; and log 1 is known to be $= 0$.

It is to be noted that a conic may be real or imaginary; the real conic, as already stated, has both real and imaginary points; an imaginary conic has, in general, only imaginary points.* Again, a conic may be proper or degenerate; a pair of intersecting straight lines is met by any straight line in two points, and is therefore classed as a conic; and similarly a pair of points, I, J, is classed as a conic, and would serve to determine two lines through any point O, viz. OI, OJ. The easiest way of forming a conception of this mode of degeneration is to consider an ellipse and hyperbola with the same axis AA', defined by the relation $PN^2 : AN.NA' = k$, where for the ellipse k is positive, and for the hyperbola negative. Letting the quantity k diminish, it is seen that the curves become flatter and flatter, approximating to the straight line, while the tangents have a tendency to pass through the points A, A'. When we take k indefinitely small, the tangents become lines through one or other of the two points A, A'; looked at in this way, the conic is said to have degenerated into these two points. But considering the description of the curve by a moving point, we see that a part of the straight line AA' is described twice; if k be positive and indefinitely small, it is the inner part AA' that is so described; if negative, the outer part. If k be regarded as actually zero, there is no such distinction, and we have to regard the line (not any particular part of it) as described twice. Thus the full description of this form of degenerate conic is that it consists of a pair of points, with the line joining them taken twice.

In considering what conic must be chosen in order that the system of measurement may agree with ordinary measurement, we note that one of the fundamental principles of Euclidean geometry is that exactly one parallel can be drawn from a point T to a line. Now a real conic as Absolute may give two real parallels, obtained by joining T to the two points in which the line cuts the Absolute; this gives a perfectly consistent system of geometry, but not the Euclidean; it is a non-Euclidean system, the geometry of Bolyai and Lobatchewsky. An imaginary conic gives no real parallel in any case; this is again a non-Euclidean system, the geometry of Riemann and others. Neither of these will serve; we have therefore to seek among the degenerate conics. The pair of straight lines gives two distinct points on every line, and therefore two parallels; the

* It may, however, have two or four real points.

pair of points, I, J, with the repeated straight line joining them, gives the necessary two lines through every point, and on every line it gives two coincident points, and consequently through a point there can be drawn precisely one parallel to a given straight line. Hence if there be any conic which as Absolute will give ordinary metric geometry, it must be of this type; moreover, the fact that parallel straight lines intersect on the straight line at infinity shows that the repeated straight line which is a part of this degenerate conic can only be the line at infinity. Hence the Absolute that we are in search of must be a pair of points on the line at infinity.

Perpendicular lines, dividing the chord of contact of tangents from their intersection harmonically, are in this case conjugate with respect to the points I, J. Now if the pairs PQ, IJ are harmonic, the quadrilateral diagram shows that as P approaches I indefinitely on one side, Q approaches it indefinitely on the other; hence if P be at I, Q is also at I, and thus any two lines meeting at one of the points I, J, satisfy the condition of perpendicularity. Regarding a circle on AB as diameter as the locus of the intersection of perpendicular lines through A, B, the fact that AI, BI, are perpendicular shows that the circle passes through I, and likewise through J. Hence the points I, J, have the property of lying on every circle; they are the points in which every circle meets the line at infinity, and are obviously imaginary.

The conclusion so far obtained is:—*If it be possible to express the ordinary Euclidean measurement by reference to the Absolute, this Absolute must be a pair of imaginary points; the line joining these points will be the line at infinity, and every circle will pass through the points.* By somewhat refined mathematical reasoning it is shown that the constants involved in the particular functions which give the numerical value for (OP, AB) and (op, ab) can be chosen so that the generalized expressions for the distance and the angle shall, in this case, become the ordinary expressions for the distance and the angle. This having been accomplished, the conditional conclusion stated above follows.

Cayley's theory of the Absolute (in a plane) is therefore that all metric properties can be expressed under the form of projective relations to a given degenerate conic; it applies also to non-Euclidean systems of geometry, these being differentiated by a different choice of the Absolute.

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LINES COMMON TO FOUR LINEAR COMPLEXES.

BY DR. VIRGIL SNYDER.

(Read at the February meeting of the Society, 1897.)

In his discussion of the invariants of one or more linear complexes, Klein* makes the statement that four such complexes have two lines in common, which become coincident when the combinant of the four complexes vanishes, but otherwise the reality of the lines is not discussed.

The corresponding criterion for spherical geometry is of value in studying the cyclides and it can be proved by geometrical methods that the spheres common to four linear spherical complexes are real when the combinant is negative. On account of the direct interpretation of the simpler invariants from one geometry into the other, one might conclude by analogy that the same law holds here, which, however, is not the case.

For convenience, transform the quadratic relation

$$P_{12}P_{34} + P_{13}P_{42} + P_{14}P_{23} = 0$$

by the transformation

$$P_{12} = x_1 + x_2, \quad P_{13} = x_3 + x_4, \quad P_{14} = x_5 + x_6,$$
$$P_{34} = x_1 - x_2, \quad P_{42} = x_3 - x_4, \quad P_{23} = x_5 - x_6, \text{ into}$$
$$x_1^2 - x_2^2 + x_3^2 - x_4^2 + x_5^2 - x_6^2 = 0.$$

Let the four given complexes be

$$(1) \qquad \psi_i \equiv a_i x_1 + b_i x_2 + c_i x_3 + d_i x_4 + e_i x_5 + f_i x_6 = 0 \; [i = 1, 2, 3, 4].$$

The invariant of ψ_i is

$$(2) \qquad A_{ii} \equiv a_i^2 - b_i^2 + c_i^2 - d_i^2 + e_i^2 - f_i^2$$

and the simultaneous invariant of ψ_i, ψ_k is

$$(3) \qquad A_{ik} \equiv a_i a_k - b_i b_k + c_i c_k - d_i d_k + e_i e_k - f_i f_k.$$

Two complexes are in involution when their simultaneous invariant vanishes; a general complex is in involution with

* "Differentialgleichungen in Liniengeometrie," *Math. Annalen*, vol. 5.

a special one ($A_{ii} = 0$) when the axis of the special complex belongs to the general one.

Let

$$Ax_1 + Bx_2 + Cx_3 + Dx_4 + Ex_5 + Fx_6 = 0 \tag{4}$$

represent all the complexes in involution with every complex of the system

$$l\psi_1 + m\psi_2 + n\psi_3 + P\psi_4 = 0. \tag{5}$$

The axes of all the special complexes of (4) will then belong to every complex of (5) and therefore to $\psi_1, \psi_2, \psi_3, \psi_4$. The simultaneous invariant of (4) and (5) must vanish for all values of l, m, n, P, which gives four equations (by (3)) of the form

$$a_i A - b_i B + c_i C - d_i D + e_i E - f_i F = 0 [i = 1, 2, 3, 4].$$

from which any four, as A, B, C, D can be solved in terms of the other two, E, F. This gives

$$A = \frac{-E\binom{a}{e} + F\binom{a}{f}}{(a)}; \quad B = \frac{E\binom{b}{e} - F\binom{b}{f}}{(b)}$$

$$C = \frac{-E\binom{c}{e} + F\binom{c}{f}}{(c)}; \quad D = \frac{E\binom{d}{e} - F\binom{d}{f}}{(d)}$$

where $(a) = (b) = (c) = (d) = \begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}$ and

$\binom{a}{e}$ is what (a) becomes when a_i has been replaced by e_i $[i = 1, 2, 3, 4]$, and similarly for the other forms.

These values for A, B, C, D are now to be substituted in (4). This gives

$$E\psi' + F\psi'' = 0, \text{ where} \tag{6}$$

$$\psi' \equiv -\binom{a}{e}x_1 + \binom{b}{e}x_2 - \binom{c}{e}x_3 + \binom{d}{e}x_4 + (a)x_5,$$

$$\psi'' \equiv \binom{a}{f}x_1 - \binom{b}{f}x_2 + \binom{c}{f}x_3 - \binom{d}{f}x_4 + (a)x_6.$$

There are two special complexes contained in (6), whose axes belong to (5). These axes are real, coincident or

imaginary according as the combinant of ψ', ψ'' is negative, zero or positive, or according as

$$\Phi \equiv [A'^{,}{}^{''}]^2 - A' A''$$

is positive, zero or negative, [A', A'', $A'^{,}{}^{''}$ invariants of ψ', ψ''].

$$A'^{,}{}^{''} \equiv -\binom{a}{e}\binom{a}{f} + \binom{b}{e}\binom{b}{f} - \binom{c}{e}\binom{c}{f} + \binom{d}{e}\binom{d}{f}$$

$$A' \equiv \binom{a}{e}^2 - \binom{b}{e}^2 + \binom{c}{e}^2 - \binom{d}{e}^2 + (a)^2$$

$$A'' \equiv \binom{a}{f}^2 - \binom{b}{f}^2 + \binom{c}{f}^2 - \binom{d}{f}^2 - (a)^2$$

After a few simple reductions,

$$\Phi = \left[\binom{a}{e}\binom{b}{f} - \binom{a}{f}\binom{b}{e}\right]^2 - \left[\binom{a}{e}\binom{c}{f} - \binom{a}{f}\binom{c}{e}\right]^2 + \left[\binom{b}{e}\binom{c}{f} - \binom{b}{f}\binom{c}{e}\right]^2$$

$$+ \left[\binom{a}{e}\binom{d}{f} - \binom{a}{f}\binom{d}{e}\right]^2 - \left[\binom{b}{e}\binom{d}{f} - \binom{b}{f}\binom{d}{e}\right]^2 + \left[\binom{c}{e}\binom{d}{f} - \binom{c}{f}\binom{d}{e}\right]^2$$

$$+ (a)^2\left[\binom{a}{e}^2 - \binom{a}{f}^2 - \binom{b}{e}^2 + \binom{b}{f}^2 + \binom{c}{e}^2 - \binom{c}{f}^2 - \binom{d}{e}^2 + \binom{d}{f}^2 + (a)^2\right].$$

This value for Φ is now to be compared with the combinant of (5).

$$\Delta \equiv \begin{vmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{12} & A_{22} & A_{23} & A_{24} \\ A_{13} & A_{23} & A_{33} & A_{34} \\ A_{14} & A_{24} & A_{34} & A_{44} \end{vmatrix}$$

Δ may be expressed as the product of the two rectangular arrys

$$\left\{\begin{matrix} a_1 b_1 c_1 d_1 e_1 f_1 \\ a_2 b_2 c_2 d_2 e_2 f_2 \\ a_3 b_3 c_3 d_3 e_3 f_3 \\ a_4 b_4 c_4 d_4 e_4 f_4 \end{matrix}\right\} \quad \left\{\begin{matrix} a_1 - b_1 \; c_1 - d_1 \; e_1 - f_1 \\ a_2 - b_2 \; c_2 - d_2 \; e_2 - f_2 \\ a_3 - b_3 \; c_3 - d_3 \; e_3 - f_3 \\ a_4 - b_4 \; c_4 - d_4 \; e_4 - f_4 \end{matrix}\right\}$$

and is developed into

$$\Delta \equiv (a)^2 - \binom{d}{e}^2 + \binom{d}{f}^2 + \binom{a}{e}^2 - \binom{a}{f}^2 + \binom{b}{c}^2 + \binom{b}{f}^2 + \binom{c}{e}^2 - \binom{c}{f}^2$$

$$+ (a_1 b_2 e_3 f_4)^2 - (a_1 c_2 e_3 f_4)^2 + (a_1 d_2 e_3 f_4)^2 + b_1 c_2 e_3 f_4)^2 - (b_1 d_2 e_3 f_4)^2 + (c_1 d_2 e_3 f_4)^2$$

The first row is the same as the coefficient of $(a)^2$ in Φ, and from determinants,

$$(a)^2_1 (c_1 d_2 e_3 f_4)^2 = \left[\binom{a}{e}\binom{b}{f} - \binom{a}{f}\binom{b}{e}\right]^2, \text{ and similarly for}$$

the other terms, then $\Delta \equiv (a)^2 \Phi$. *The two lines common to four linear complexes are real, coincident or imaginary according as the combinant of the complexes is positive, zero or negative.* This criterion may be used to find the reality of the lines cutting four given ones, by making the principal diagonal of Δ vanish, and for A_a, the polar of the lines i, k.

CORNELL UNIVERSITY,
January 8, 1897.

THE CUBIC RESOLVENT OF A BINARY QUARTIC DERIVED BY INVARIANT DEFINITION AND PROCESS.

BY PROFESSOR H. S. WHITE.

(Read before the Conference at Chicago, January 1, 1897.)

In the usual discussion of a binary quartic the cubic resolvent first arises as an auxiliary in factoring the quantic, or what is the same thing, in reducing it to a determinate normal form. The coefficients of this cubic, when found, prove to be rational invariants of the original quartic. Such a fact appears as a surprise, since the invariant character of the roots of the cubic is rarely made prominent at

the beginning of the inquiry. As a supplement to such discussions, the following derivation of the resolvent cubic has the merit of employing invariant definitions and processes throughout. It might properly precede the application of this cubic to the reduction or solution of the quartic.

As a preliminary step, consider two quadrics: $P = p_x^2$ and $Q = q_x^2$, together with their first transvectant: $A = a_x^2 = (P, Q) = (pq)\, p_x q_x$. The form A has then this property (completely defining it) that its second transvectant with either of the quadrics P and Q vanishes identically. Each of these three quadrics, equated to zero, is represented geometrically by two points of a line, and the two points A divide harmonically both the pair P and the pair Q. These well known relations are proved by the identities:

$$(P, A)_2 = (p'a)^2 = (pq)\,(p'p)\,(p'q) = \tfrac{1}{2}(p'p)^2\,(qq) \equiv 0,$$

$$(Q, A)_2 = (q'a)^2 = (pq)\,(q'p)\,(q'q) = \tfrac{1}{2}(q'q)^2\,(pp) \equiv 0.$$

The form a_x^2 satisfies these two invariant equations rational in the coefficient of p_x^2 or q_x^2. But further, if we consider the quartic composed of the quadric factors p_x^2 and q_x^2:

$$f = a_x^4 = p_x^2 \cdot q_x^2,$$

the form a_x^2 satisfies a peculiar invariant equation which is rational in the coefficients of this quartic f and one adjoined quantity λ. *The second transvectant, namely, of quartic and quadric is a constant multiple of the quadric a_x^2 itself.* The proof is as follows:

$$4(f, A)_2 = 4(aa)^2 a_x^2 =$$

$$= (pq)\{(pp')\,(qp')\,q_x^2 q_x' + (pq')\,(qq')\,p_x^2 p_x' +$$

$$(pp')\,(qq')\,p_x' q_x' + (pq')\,(qp')\,p_x' q_x'\}.$$

Of these four terms the first two vanish identically, as was seen above; the third reduces thus:

$$(pp')(qq')(pq)\,p_x' q_x' = \tfrac{1}{2}(pp')^2 (qq')\, q_x q_x' \equiv 0;$$

the fourth becomes, by similar reduction,

$$-\tfrac{1}{2}(pq)^2\,(p'q')\,p_x' q_x' = -\tfrac{1}{2}(pq)^2 a_x^2,$$

the quadric a_x^2 multiplied by a constant $-\tfrac{1}{2}(pq)^2$.

Here we may begin *de novo* and inquire: having given a quartic a_x^4, how many quadrics a_x^2 can be found to satisfy identically a transvectant reproductive equation

$$(a\,a)^2\,a_x^2 \equiv \lambda \cdot a_x^2,$$

and what is the equation determining λ? This question is invariant in form, showing that the constants λ and the corresponding quadrics a_x^2, if determinate, must be invariants and covariants of the quartic a_x^4.

The characteristic equation is quadric, and yields three independent linear equations for the a's. Hilbert has eliminated from these the co-efficients of a_x^2 and reduced the resulting cubic determinant in λ.* But the same end may be accomplished without leaving the domain of invariant equations, by using an identity elegantly proved by Clebsch. Repeated use of the defining equations gives the series of relations all equivalent to the first:

$$(a\,a)^2\,a_x^2 = \lambda \cdot a_x^2,$$
$$(a\,a)^2(a\,b)^2\,b_x^2 = \lambda^2 \cdot a_x^2,$$
$$(a\,a)^2(a\,b)^2(b\,c)^2\,c_x^2 = \lambda^3 \cdot a_x^2, \text{ etc.}$$

Using Clebsch's notation, call P the operation of deriving $(a\,a)^2\,a_x^2$ from a_x^2—

$$P(a_x^2) = (a\,a)^2 a_x^2;$$

then his identical relation is

$$P^3(a_x^2) \equiv \frac{i}{2} \cdot P(a_x^2) + \frac{j}{3} \cdot a_x^2,\dagger$$

i and j denoting the fundamental invariants of the quartic:

$$i = (a\,b)^4, j = (a\,b)^2(b\,c)^2(c\,a)^2.$$

By virtue of the above relations we have, however:

$$\left(P^3 - \frac{i}{2}P - \frac{j}{3}\cdot\right)(a_x^2) = \left(\lambda^3 - \frac{i}{2}\cdot\lambda - \frac{j}{3}\right)\cdot a_x^2 = 0.$$

Hence, if a_x^2 be a determinate form, we must have λ a root of the equation

$$\lambda^3 - \frac{i}{2}\lambda - \frac{j}{3} = 0,$$

the reducing cubic.

Looking back to the point where λ was first introduced, we see it denoting an irrational invariant $-\frac{1}{8}(p\,q)^2$ of the quartic. There are, in general, as there must be in ac-

* *Math. Annalen*, Bd., 28, p. 433.
† *Binäre Formen*, p. 219.

cordance with this fact, three possible values of λ, and to each a corresponding quadric linearly determined.

A known invariant identity has yielded readily the determining equation for the irrational invariant λ. I intend at another time to show that the reverse process is legitimate, direct and fruitful.

NORTHWESTERN UNIVERSITY,
January, 1897.

TWO BOOKS ON ELEMENTARY GEOMETRY.

Elements of Geometry. By ANDREW W. PHILLIPS, PH.D., and IRVING FISHER, PH.D., Professors in Yale University. New York, Harper and Bros. 1896. 540 pp. $1.75.

Elementary Solid Geometry and Mensuration. By HENRY DALLAS THOMPSON, D.Sc., PH.D., Professor of Mathematics in Princeton University. New York and London, The Macmillan Company. 1896. vii+199 pp. $1.25.

Among the multitude of books on elementary geometry brought out within the last few years, two that have appeared recently rise above the general level and call for notice.

The larger and more ambitious of these—"Elements of Geometry," by Phillips and Fisher, a work of over 500 pages—covers the ground of the ordinary school text-books on plane and solid geometry, and makes the rough places smooth before the pupil with all the modern equipments in the way of realistic diagrams and conspicuous and varied types. We are tempted to wonder whether by thus diminishing the necessity for the exercise of the imagination and the attention, we may not lose some of the qualities which render geometry invaluable as a mental discipline. A noteworthy innovation is the final chapter, headed an "Introduction to Modern Geometry," which gives a brief account—so brief as to be scarcely more than a summary—of inversion, linkages, stereographic projection, poles and polars, the nine-points circle, perspective, duality, involution, antiparallels and the axioms of plane, spherical and pseudospherical geometries.

The chief novelties to be noted in the first part of the book are the early introduction of the idea of symmetry and the use of rotation about a point to prove theorems, relating to parallel lines. The note on p. 23, designed ap-

parently to teach the beginner how far he may put his trust in axioms, shows a fine disregard of philosophic doubts. It says: "Lobatchewsky in 1829 proved that we can never get rid of the parallel axiom without assuming the space in which we live to be very different from what we know it to be through experience. Lobatchewsky tried to imagine a different sort of universe in which the parallel axiom would not be true. This imaginary kind of space is called *non-Euclidean* space, whereas the space in which we really live is called *Euclidean*, because Euclid (about 300 B. C.) first wrote a systematic geometry of our space."

The treatment of limits is clear and accurate and the definitions and proofs are, for the most part, well-worded and concise. The use of a strange assortment of small letters—generally m, s, w—to denote sometimes angles, sometimes lines, is objectionable; and such formulæ as those for the area of a triangle and the radius of its circumcircle in terms of the sides should hardly, if given at all, be placed among the examples.

On turning to the "Geometry of Space" the diagrams delight the eye. Each theorem is illustrated by a photogravure of a model constructed for the purpose of demonstrating it, and side by side with each of the photogravures is a simplified drawing showing merely the lines necessary to the demonstration. The book will help us to wait for that millennium when every teacher of solid geometry shall have models ready to his hand.

As regards the subject matter of this division the most obvious criticism is that there is too much of it; 120 theorems are collected and many of these, those, for instance, on radially situated polyedra, on spherical polygons and on the construction of the dodecaedron and icosaedron might well be spared, or relegated to the oblivion of an appendix. There is a lack of the gradation in difficulty that is desirable in a text-book for young pupils, while the more mature student should be led on to principles rather than encouraged to dally over isolated theorems. In many cases generalization has been carried too far and the most important facts, *e. g.*, the expression for the volume of a sphere, stated merely as corollaries.

The third section, on Modern Geometry, is too curtailed, it is to be hoped that in a future edition it may be expanded; remarks on space of constant curvature are of little value before curvature itself has been defined. The indices are useful and the book is well printed. The absence of any mention of conic sections is unexplained.

In direct opposition to the intuitional method of Drs. Phillips and Fisher is the logical form of Dr. Thompson's "Elementary Solid Geometry and Mensuration;" to quote from the preface, at the risk of being suspected of having read no further, "the attempt has been made in every case to secure the belief in the truth of the proposition from previous geometrical knowledge, rather than from any direct generalizations from material bodies." This attempt has been carried out in an able manner, and the result is a mathematical treatise sustaining a continued argument, instead of the arbitrarily arranged collection of theorems too often met with. The theorems given, though simple and few in number, are well chosen and sufficient to serve as an introduction to the higher branches of mathematics.

The straight line is frequently a source of difficulty to writers on elementary geometry. Many who have steered clear of the Scylla of a definition have fallen into the Charybdis of a postulate which fails to state the very property subsequently assumed as fundamental.

Dr. Thompson's postulate is as follows:

"There is in space a line, called the straight line, possessing the following properties:

(a) Any segment can be moved forward or backward in the line without flexing it.

(b) Through any two points there is one, and but one, such line." To which, he says, *may* be added "(c) A segment of it is a shortest line in space between *any* two points."

Disregarding the ambiguously worded shortest distance property a circle through a fixed point satisfies all the requirements formulated in (a) and (b), and yet a page or two further on (Cor. 3, § 18) we find the statement that "any straight line of the one plane may be brought into coincidence with any straight line of the other."

Other instances of similar slackness are, however, hard to find; and such theorems as "if two planes meet, they meet in at least two points" and "a pair of perpendiculars to the axis of a dihedral, one in each of the two faces, through a point on the axis, makes an angle which is the same, no matter where the point is taken on the axis," which are too often left to intuition, are here rigorously proved. The systematic use of capitals for points, small letters for lines and Greek letters for planes is a detail, but admirable, and the book throughout shows a scholarly treatment of an elementary subject.

ISABEL MADDISON.

BRYN MAWR COLLEGE, *February*, 1897.

SHORTER NOTICES.

Index operum Leonardi Euleri, confectus a IOANNE G. HAGEN, S. J., Directore speculæ astronomicae Collegii Georgiopolitani, Washington, D. C. Berolini, Dames, 1896. 8vo. viii+80 pp.

This bibliography of Euler's works contains nearly 800 titles, distributed in four main groups: mathematical, physical, astronomical and miscellaneous. Each group is again subdivided by subjects. The titles are derived from previous bibliographies, but have been carefully collated and corrected. In an appendix we find the titles of some unpublished papers, of some that seem to be lost, and of some that have been erroneously attributed to Euler. The author undertook this work in the hope that it may serve as basis for a complete edition of Euler's works. It is estimated that this would require 25 quarto volumes, of 640 pages each, and that the expense per volume would be about $1,500. There can be no doubt that such an edition would be very valuable. It is, therefore, to be hoped that the learned author of the "Synopsis der höheren Mathematik" may succeed in securing the financial means for this gigantic undertaking. ALEXANDER ZIWET.

Briot and Bouquet's Elements of Analytical Geometry of Two Dimensions. Fourteenth Edition. Translated and edited by JAMES HARRINGTON BOYD, Instructor in Mathematics in the University of Chicago. Chicago and New York, Werner School Book Company, [1896]. 8vo. iii+581 pp.

The "Leçons de Géométrie Analytique" of Briot and Bouquet has been long recognized as one of the most successful text-books ever written. The fourteenth edition of the work, which was revised and annotated by Professor Appell, was described in the first volume of the BULLETIN OF THE NEW YORK MATHEMATICAL SOCIETY by Dr. C. H. Chapman under the title, "A French Analytical Geometry." The desirability of an English translation has been so often commented upon that it is surprising that we have had to wait for one until now. The translation, which seems to be fairly good, gives only that part of the original which relates to plane geometry, or somewhat more than two-thirds of the whole work. The translator nowhere gives the name of the French editor or refers to the fact that the translation represents but a part of the French work from which it is taken. THOMAS S. FISKE.

NOTE ON THE HISTORY OF THE MAP-COLORING PROBLEM.

THE map-coloring problem has recently been recalled to mind by some questions in the *Intermédiaire des Mathématiciens* and a paper by Mr. P. J. Heawood in the *Quarterly Journal of Mathematics*. Judging from the references given in these places, which mention merely Cayley and De Morgan among the earlier writers on the subject, it does not seem to be generally known that Möbius discussed the question, in a slightly different form, in his lectures in 1840. A note by Professor Baltzer, entitled, "Eine Erinnerung an Möbius und seinen Freund Weiske," relates how the problem was propounded to Möbius by his friend, Professor Weiske, the philologist, and is to be found in the *Berichte der Sächsischen Gesellschaft der Wissenchaften zu Leipzig*. Mathematisch-physiche Classe. Bd. 37, 1885, pp. 1–6.

ISABEL MADDISON.

BRYN MAWR COLLEGE.
February, 1897.

NOTES.

A REGULAR meeting of the AMERICAN MATHEMATICAL SOCIETY was held in New York on Saturday afternoon, March 27, at three o'clock, the Vice-President, Professor R. S. WOODWARD, in the chair. There were seventeen members present. On the recommendation of the Council, the following persons, nominated at the preceding meeting, were elected to membership: Mr. WILLIAM FREELAND, Harvard School, New York, N. Y.; Mr. GILBERT HARRISON JOHN HURST, Eton College, Windsor, England. Four nominations for membership were received. The amendments to the Constitution and By-Laws recommended by the Council at the preceding meeting were unanimously adopted.

The following papers were read:—

(1) Mr. M. B. PORTER: "On the roots of the hypergeometric series."

(2) Mr. JOSEPH COTTIER: "On the expression of the general equations of hydrodynamics in terms of curvilinear coordinates."

(3) Professor JAMES P. PIERPONT: "On modular equations."

THE Council has authorized the organization of a Section of the AMERICAN MATHEMATICAL SOCIETY by those members who may be present at the conference to be held in Chicago on April 24.

THE local committee of the International Congress of Mathematicians to be held at Zürich announces the following general programme: Meetings of the entire Congress will be held on Monday, August 9, and Wednesday, August 11, at which questions of a more general character will be discussed. Papers dealing with special subjects will be presented before the various sections on Tuesday, August 10. The Congress will direct its attention not only to purely scientific questions, but also to matters of an executive and business nature, such as questions of bibliography, lexicography, terminology, coöperative scientific undertakings, including historical investigations, comprehensive reports, the publication of treatises, the holding of expositions, etc. Discussions concerning the relation of mathematics to other branches of science, to the technic arts, to practical life, etc., would also be appropriately included in the proceedings.

PROFESSOR KARL WEIERSTRASS died at Berlin on February 19, aged eighty-one years.

PROFESSOR JAMES JOSEPH SYLVESTER died at London on March 15, aged eighty-three years.

PROFESSOR W. W. HENDRICKSON, of the department of of mathematics of the U. S. Naval Academy, has been appointed Superintendent of the American Ephemeris and Nautical Almanac, in succession to Professor SIMON NEWCOMB, who, having reached the age limit fixed by law, was retired on March 12.

UNIVERSITY OF BERLIN. The mathematical courses announced for the summer semester are the following:—By Professor FUCHS: Applications of elliptic functions; On the representation of functions which are defined by algebraic differential equations.—By Professor SCHWARZ: Arithmetical theory of complex quantities; Differential calculus; Theory of analytic functions; Mathematical colloquium.—By Professor FROBENIUS: Theory of algebraic equations (second part).—By Professor HETTNER: On approximate calculation of definite integrals.—By Professor KNOBLAUCH: Integral calculus; Selected chapters of the theory of elliptic functions; Analytical mechanics.—By

Professor HENSEL: Determinants; On complex multiplication of elliptic functions; Analytic and algebraic methods for complex multiplication of elliptic functions.—By Professor BAUSHINGER: Introduction to celestial mechanics.—By Professor LEHMANN-FILHÉS: Celestial mechanics; Determination of orbits of double stars.—By Professor PLANCK: Exercises in mathematical physics.—By Professor VON BEZOLD; Theoretical meteorology (Statics and dynamics of the atmosphere).—By Dr. HOPPE: Integral calculus; Analytical mechanics.—By Dr. KÖTTER: Analytical geometry of systems and complexes of rays; Kummer's surfaces with sixteen nodes; On parallel and central projection.—By Dr. SCHLESINGER: Theory and applications of definite integrals; Analytical geometry of two and three dimensions.—By Dr. KRIGAR-MENZEL: Hyperbolic functions.—By Dr. GLAN: Quaternions; Elements of theoretical physics; Theory of heat.

HARVARD UNIVERSITY. The following advanced mathematical courses are offered for the year 1897–98:—By Professor J. M. PEIRCE: Quaternions (first course); Surfaces of the First and Second Degrees; The Algebra of Logic†.—By Professor ASAPH HALL (U. S. Navy): The Theory of Planetary Motions.—By Professor BYERLY: Modern Geometry; Rigid Dynamics†.—By Professors BYERLY and B. O. PEIRCE: Fourier's Series, Spherical Harmonics and Potential Function.—By Professor B. O. PEIRCE: Hydrodynamics.—By Professor OSGOOD: Differential and Integral Calculus (second course); Infinite Series and Products†; Elliptic Integrals and Functions†.—By Professor BÔCHER: Theory of Equations and Invariants†; Higher Algebra†; Theory of Functions (first course); Linear Differential Equations.

These courses will each consist of three lectures a week throughout the entire academic year except those marked† which consist of about half this number of lectures. The following courses for Reading and Research are also offered:—By Professor J. M. PEIRCE: Elliott, Algebra of Quantics.—By Professor HALL: The Plasticity of the Earth.—By Professor BYERLY: Picard, Traité d'Analyse Vol. I.—By Professor B. O. PEIRCE: Methods in Mathematical Physics.—By Professor OSGOOD: The Elliptic Modular Functions.—By Professor BÔCHER: Topics in Linear Differential Equations.

A Mathematical Conference will meet twice a month.

The more elementary subjects taught will be the same as this year (see BULLETIN for May, 1896).

THE fifth volume of HELMHOLTZ's "Lectures on theoretical physics" has just appeared; it contains the electromagnetic theory of light. This volume will be followed in order by Part 2 of Vol. I: Dynamics of discrete mass-points, edited by O. Krigar-Menzel; Part 1 of Vol. I: The general principles of the physical sciences, edited by A. König; and Vol. III: Mathematical principles of acoustics, edited by A. König and C. Runge. At a later period may be expected Vol. II: Dynamics of continuous masses, Vol. IV: Electrodynamics and theory of magnetism, and Vol. VI: Theory of heat, all to be edited by O. Krigar-Menzel. L. Voss, of Hamburg, is the publisher.

JOHN WILEY AND SONS have in press a work on the elements of the theory of functions, by Professor A. S. CHESSIN, of the Johns Hopkins University.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

APPELL (P.) et LACOUR (E.). Principes de la théorie des fonctions elliptiques et applications. Paris, Gauthier-Villars, 1896. 8vo. 9 and 421 pp. Fr. 12.00

BASS (E. W.). Differential calculus. New York, Wiley, 1896. 12mo. $4.00

BOHLMANN (G.). See SERRET (J. A.).

BONNEL (J.). Les hypothèses dans la géométrie; éléments de la théorie atomique. Paris, Gauthier-Villars, 1896. 8vo. Fr. 1.50

BOUQUET. See BRIOT and BOUQUET.

BOYD (J. H.). See BRIOT and BOUQUET.

BRIOT and BOUQUET. The elements of analytical geometry of two dimensions. 14th edition. Authorized English translation by J. H. Boyd. Chicago, Werner School Book Co., 1896. 8vo. 582 pp. Cloth. $2.00

DICKSTEIN (S.). See SCHLEGEL (V.).

EPAPHRODITUS et VITRUVIUS. Traités d'arpentage et de géométrie. Nouveau texte, publié d'après le manuscrit latin 13,084 de la Bibliothèque Royale de Munich par V. Mortet. Avec introduction par P. Tannery. Paris, Gauthier-Villars, 1896. 4to. 4 and 44 pp. avec 2 planches et figures.

FRICKE (R.). Hauptsätze der Differential- und Integralrechnung. Als Leitfaden zum Gebrauch bei Vorlesungen zusammengestellt. (2 Theile.) Theil I. Braunschweig, Vieweg, 1897. 8vo. 9 and 80 pp. Mk. 2.00

FURTWÄNGLER (P.). Zur Theorie der in Linearfaktoren zerlegbaren, ganzzahligen ternären cubischen Formen. [Diss.] Göttingen, Vandenhœck, 1897. 8vo. 63 pp. Mk. 1.60

GALDEANO (Z. G. DE). Las modernas generalizaciones expresadas por el algebra simbolica, les geometrias no Euclideas, etc. Madrid, 1896. 8vo. 142 pp. Mk. 3.00

HARNACK (A.). See SERRET (J. A.).

HESSE (L. O.). Gesammelte Werke. Herausgegeben von der mathematisch-physikalischen Classe der königlich bayerischen Akademie der Wissenschaften. München, Franz, 1897. 4to. 8 and 732 pp. Portrait. Mk. 24.00

KIEPERT (L.). Grundriss der Differential- und Integral- Rechnung. 6te vollständig umgearbeitete und vermehrte Auflage des gleichnamigen Leitfadens von M. Stegemann. (In 2 Theilen.) Theil II. Integral-Rechnung. Hannover, Helwing, 1896. 8vo. 18 and 599 pp. Mk. 11.50

KILBINGER (G.). Der Axencomplex der Rotations-Flächen zweiter Ordnung. Mühlhausen, 1896. 4to. 13 pp. Mk. 1.20

KLEIN (F.). Ausgewählte Capitel der Zahlenlehre. Zweistündige Vorlesung im Winter 1895–96 und Sommer 1896. (Autographirte Vorlesungshefte, III.) To be obtained from Dr. F. Schilling, Aachen, Germany.

LACOUR (E.). See APPELL (P.).

LONEY (S. L.). The elements of coördinate geometry. 2d edition, revised. London and New York, Macmillan, 1897. 8vo. 6s.

MANSION (P.). Notice sur les travaux mathématiques de E. C. Catalan. Paris, Gauthier-Villars, 1896. 12mo. 60 pp. Fr. 1.00

MANSION (P.). Premiers principes de la métagéométrie ou géométrie générale. Paris, Gauthier-Villars, 1896. 8vo. Fr. 2.00

MORTET (V.). See EPAPHRODITUS et VITRUVIUS.

SAUSSURE (R. DE). Principles of a new line-geometry. (*Catholic University Bulletin*, January, 1897, Vol. III, No. 1, pp. 81–86.) 8vo.

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